Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 3, pp. 393–402, 1994

0021–8928/94 \$24.00+0.00

0021–8928(94)00064-6

ASYMPTOTIC ANALYSIS OF DYNAMICAL SYSTEMS SUBJECTED TO HIGH-FREQUENCY INTERACTIONS†

L. D. AKULENKO

Moscow

(*Received* 13 *January* 1993)

A class of dynamic objects of general form subjected to rapidly changing, and, in particular, high-frequency quasiperiodic external interactions is investigated. Conditions under which the system of equations of motion can be reduced to standard form are obtained. A transformation which allows an asymptotic analysis to be made using methods of separation of motion (the averaging method) which generalizes existing transformations is realized. In the first approximation the corresponding system is obtained and the autonomous system for slow displacements is studied qualitatively. The approach is illustrated by solving a number of problems for a system with one degree of freedom and variable parameters. Systems such as a non-linear oscillator and a simple pendulum are considered. External torques, kinematic excitation by vibrations of the point of suspension and parametric excitation by changing the length of the pendulum are taken as the high-frequency periodic interactions. Other models are considered.

1. INITIAL ASSUMPTIONS AND FORMULATIONS OF THE PROBLEM

We will consider a non-linear dynamical system of fairly general form. We shall assume that the mechanical object under investigation is acted upon by rapidly changing external forces, in particular high-frequency ones. This will be signified by a scalar parameter λ such that $\lambda \gg 1$ after transforming the system to dimensionless variables and parameters. To fix our ideas, we consider the Cauchy problem

$$X^{\cdot\cdot} = F(\lambda t, X, X^{\cdot}; \lambda), \quad X(t_0) = X^0, \quad X^{\cdot}(t_0) = V^0 \tag{1.1}$$

with a large parameter λ.

Here X, $X^{\cdot} = dX/dt$, and F are vectors of arbitrary dimensions $n \geq 1$, and t_0, X^0, V^0 are the initial data for t, X, $X^{\cdot}$, respectively. the large parameter λ characterizes the relative rate of change of external interactions (forces, system parameters, etc.). The details of the structure and the smoothness of F will be specified below after certain transformations of (1.1).

We introduce the fast time $\theta = \lambda t$. Problem (1.1) is then transformed into the following Cauchy problem with a small parameter $\varepsilon = \lambda^{-1} > 0$ (see [1–3])

$$\begin{gathered} x'' = f(\theta, x, x', \varepsilon), \quad \theta \geq \theta_0, \quad x(\theta_0) = x^0, \quad x'(\theta_0) = \upsilon^0 \\ x(\theta) \equiv X(t), \quad x^0 = X^0, \quad \upsilon^0 = \varepsilon V^0, \quad f \equiv \varepsilon^2 F(\theta, x, \varepsilon^{-1} x', \varepsilon^{-1}) \end{gathered} \tag{1.2}$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 3, pp. 23–31, 1994.

In what follows we will consider (1.2) under the following assumptions regarding the properties of f

$$f(\theta, x, x', \varepsilon) \equiv \varepsilon p(\theta, x, x') + \varepsilon^2 q(\theta, x, \varepsilon^{-1}x', \varepsilon) \tag{1.3}$$

Here p and q are either bounded and continuous as functions of the argument $\theta \geqslant \theta_0$ or admit of uniform averages with respect to θ. This requirement will be refined and relaxed below. It is assumed that p and q are sufficiently smooth and regular as functions of x in a bounded open domain $x \in D_x$. It is also assumed that p is smooth as a function of the derivative $\upsilon = x'$ in an ε-neighbourhood of $\upsilon = 0$, while q is defined and smooth as a function of $V = \varepsilon^{-1}\upsilon$ in a bounded open domain $V \in D_V(\operatorname{diam} D_V \sim 1)$. Moreover, q is assumed to be a continuous function of $\varepsilon \in [0, \varepsilon_0]$.

In what follows the Cauchy problem (1.2), (1.3) will be examined in the asymptotically large interval $\theta - \theta_0 \in [0, \Theta\varepsilon^{-1}]$ of the fast argument, where $\Theta \sim 1$. In the general case it is difficult to use the averaging method because the system fails to be of the standard form (according to Bogolyubov's terminology [3]). the reduction of a system with rapidly rotating phase [3, 4] of the form $z^{\cdot} = g(\alpha, z)$ with $\alpha^{\cdot} \sim \lambda$, where $z = (x, \upsilon)$ and $\upsilon = x$, to (1.2) gives the expression $f \equiv \varepsilon^2 q$ (without the term εp). We remark that q may be regular as a function of $V = \varepsilon^{-1}\upsilon$ and υ in the appropriate domains. When q is a smooth function of ε, this follows directly from (1.3).

The study of the dynamics of mechanical objects subjected to rapidly changing kinematic interactions and forces leads to systems such as (1.2), (1.3). For example, see the problem of a pendulum with a vibrating point of suspension [2–6]. The periodic motion of oscillatory or rotationally oscillatory systems (1.2), (1.3) with $q \equiv 0$, i.e. $x'' = \varepsilon p(\theta, x, x')$, has been analysed by the Lyapunov–Poincaré methods [1, 2]. Constructive conditions for the existence, uniqueness, and stability of those periodic motions $x(\theta, \varepsilon)$ that turn into $x_0 = \upsilon^*\theta + x^*$ for $\varepsilon = 0$, where υ^*, x^* are certain constants, have been established. For the oscillatory components x_i of x, we have $\upsilon_i^* = 0$. The function p must be periodic with respect to the rotational components x_j. We remark that the averaging method enables us to study such a system during a relatively short time interval $\theta - \theta_0 \sim \varepsilon^{-1/2}$ only (with a small parameter $\varepsilon^{1/2}$). In the first approximation the corresponding averaged system will have the form $x'' = p_0(x)$ in terms of the slow time $\tau = \varepsilon^{1/2}\theta$, where $p_0(x) \equiv \langle p(\theta, x, 0)\rangle_\theta$ denote the mean values with respect to θ. If $p \equiv 0$ in (1.3), which corresponds to the well-known case of rapidly rotating phase [3, 4], then the first approximation of the averaged system for (1.2), (1.3) has the form $x'' = q_0(x, x')$, where the prime denotes a derivative with respect to the slow time $\tau = \varepsilon\theta$ and $q_0(x, x') = \langle q(\theta, x, x', 0)\rangle_\theta$ is the average with respect to θ. We remark that the above system for $z = (x, x^{\cdot})$ with rapidly rotating phase $\alpha^{\cdot} = \lambda\omega(z) + A(\theta, z)$ can be reduced to the form (1.2), (1.3) with $p \equiv 0$.

We will now study the asymptotic behaviour of system (1.2), (1.3), which generalizes the previous ones. By analysing the approach developed in [3] to solve the problem of the oscillations of a pendulum with a vertically vibrating point of suspension, one can generalize the change employed and, when additional conditions for $p(\theta, x, x')$ are satisfied reduce the system to stationary form with parameter ε.

2. REDUCTION OF THE SYSTEM TO STANDARD FORM

The additional condition

$$p_0(x) = \langle p(\theta, x, 0)\rangle_\theta \equiv 0, \quad x \in D_x \tag{2.1}$$

is imposed on $p = p(\theta, x, \upsilon)$.

This equality has a well-defined mechanical meaning. If the velocity $\upsilon = 0$ at some instant, then the mean increment due to p vanishes. When (2.1) is satisfied, the function $p^*(\theta, x)$, defined by integrating $p(\theta, x, 0)$, is uniformly bounded or admits of a uniform average with respect to θ. We have

$$p^*(\theta, x) = \int p(\theta, x, 0)d\theta, \quad |p^*| \leq C, \quad \theta \geq \theta_0, \quad x \in D_x \tag{2.2}$$

where $C = \text{const}$. It is convenient to integrate from θ_0 to θ in (2.2). These or other limits of integration are omitted for brevity. By analogy with $p_0(x)$ in (2.1), we introduce $p_0^*(x)$, which can be obtained by averaging $p^*(\theta,x)$ in (2.2) over θ

$$\overset{*}{p}_0(x) = \langle p^*(\theta, x)\rangle_\theta, \quad x \in D_x \tag{2.3}$$

Using the functions p^* and p_0^* given by (2.2) and (2.3), we define a function $p^{**}(\theta, x)$ which is uniformly bounded or admits of uniform averages and can be constructed in the same way as p^* in (2.2)

$$p^{**}(\theta, x) = \int [p^*(\theta, x) - \overset{*}{p}_0(x)]d\theta \tag{2.4}$$

We will now use the above functions p^*, p_0^*, p^{**} given by (2.2)–(2.4) to replace the original variable x and $\upsilon = x'$ in (1.2), (1.3) by new variables y and u as follows:

$$x = y + \varepsilon p^{**}(\theta, y), \quad \upsilon = \varepsilon u + \varepsilon[p^*(\theta, y) - \overset{*}{p}_0(y)] \tag{2.5}$$

We differentiate the expression for x with respect to θ and equate it to υ in accordance with (2.5). Solving the linear equation for y' we get

$$\begin{aligned} &y' = \varepsilon Y(\theta, y, u, \varepsilon) \\ &Y(\theta, y, u, \varepsilon) \equiv [I + \varepsilon P^{**}(\theta, y)]^{-1}u, \quad P^{**} = \partial p^{**}/\partial y \end{aligned} \tag{2.6}$$

Here I is the unit $(n \times n)$-matrix and Y is a function admitting of a uniform average over θ. The latter can be represented as $Y = (1 - \varepsilon P^{**} + \varepsilon^2 p^{**2} - \dots)u$. We note that the replacement (2.5) leads to an equation for y, the right-hand side of which is linear with respect to u with $y' = \varepsilon u + O(\varepsilon^2)$.

We differentiate the expression for υ in (2.5) with respect to θ using (1.2), (1.3), (2.2)–(2.5), and expression (2.6) obtained for y'

$$\begin{aligned} &\upsilon' = \varepsilon u' + \varepsilon p(\theta, y, 0) + \varepsilon^2[P^*(\theta, y) - \overset{*}{P}_0(y)]Y(\theta, y, u, \varepsilon) = \\ &= \varepsilon p(\theta, y + \varepsilon p^{**}, \quad \varepsilon u + \varepsilon(p^* - \overset{*}{p}_0)) + \varepsilon^2 q(\theta, y + \varepsilon p^{**}, u + p^* - \overset{*}{p}_0, \varepsilon) \end{aligned} \tag{2.7}$$

Here the dependence of the known functions p^*, p_0^*, p^{**} on θ and y is omitted for brevity. The $(n \times n)$ square matrices $P^*(\theta, y)$ and $P_0^*(y) = \langle P^*(\theta, y)\rangle_\theta$ can be defined in the same way as $P^{**}(\theta, y)$, namely $P^* = p_y^{*\prime}$ and $P_0^* = p_{0y}^{*\prime}$.

On dividing by $\varepsilon > 0$, Eq. (2.7) can be solved in an elementary way for u'

$$\begin{aligned} &u' = \varepsilon U(\theta, y, u, \varepsilon) \\ &\varepsilon U \equiv p(\theta, y + \varepsilon p^*, \varepsilon u + \varepsilon(p^* - \overset{*}{p}_0) - p(\theta, y, 0) - \varepsilon(P^* - \overset{*}{p}_0)Y + \varepsilon q(\theta, y + \varepsilon p^{**}, u + \\ &+ p^* - \overset{*}{p}_0, \varepsilon) = \varepsilon P_x p^{**} + \varepsilon P_\upsilon(u + p^* - \overset{*}{p}_0) - \varepsilon(P^* - \overset{*}{p}_0)u + \varepsilon q(\theta, y, u + \\ &+ p^* - \overset{*}{p}_0, 0) + \varepsilon^2 \dots, \quad P_{x,\upsilon} = p'_{x,\upsilon}(\theta, y, 0) \end{aligned} \tag{2.8}$$

Note that the right-hand side of (2.8) turns out to be a non-linear function of u even in the first approximation in ε because of the assumed dependence of q on $\varepsilon^{-1}\upsilon$.

Thus one obtains the system (2.6), (2.8) of $2n$ equations with the new unknowns y and u, which is of standard form [3–6]. The initial values $y(\theta_0)$ and $u(\theta_0)$ can be obtained from (2.5). We will assume that the integration in (2.2) and (2.4) is carried out in the limits from θ_0 to θ. Then

$$y(\theta_0) = y^0 = X^0, \quad u(\theta_0) = u^0 = V^0 \tag{2.9}$$

It follows that the desired standard system of equations with parameter ε and the initial conditions are completely defined by (2.6), (2.8) and (2.9). It can be subjected to further analysis based on developed methods of separation of variables ([3–6], etc.).

The above scheme for reduction to standard form can be generalized and extended to a system of the form

$$X^{\cdot\cdot} = F(\lambda t, X, X^{\cdot}, Z; \lambda), \quad Z^{\cdot} = G(\lambda t, X, X^{\cdot}, Z; \lambda) \tag{2.10}$$

(Z is a vector of arbitrary dimension $m \geqslant 0$ and the initial values X^0, V^0, Z^0 are given). To this end, in the same way as above, we introduce in (2.10) the fast time $\theta = \lambda t$, the functions $x(\theta) \equiv X(t)$ and $z(\theta) \equiv Z(t)$, and the following representations of the right-hand sides (the prime denotes a derivative with respect to θ)

$$\begin{aligned} &f(\theta, x, x', z, \varepsilon) \equiv \lambda^{-2} F(\theta, x, \lambda x', z; \lambda), \quad \varepsilon = \lambda^{-1} \\ &\varepsilon g(\theta, x, \varepsilon^{-1} x', z; \varepsilon) \equiv \lambda^{-1} G(\theta, x, \lambda x', z; \lambda) \end{aligned} \tag{2.11}$$

As a result we obtain a dynamical system of the form (1.2), (1.3) with a "slowly changing parameter" [3, 4]

$$\begin{aligned} &x' = \upsilon, \quad \upsilon' = f(\theta, x, \upsilon, z, \varepsilon), \quad z' = \varepsilon g(\theta, x, \varepsilon^{-1}\upsilon, z, \varepsilon) \\ &f \equiv \varepsilon p(\theta, x, \upsilon, z) + \varepsilon^2 q(\theta, x, \varepsilon^{-1}\upsilon, z, \varepsilon) \end{aligned} \tag{2.12}$$

Using a replacement of the type (2.5) from x, υ to y, u, which also depends on $h (z \to h)$, we obtain the equations in standard form

$$\begin{aligned} &y' = \varepsilon Y(\theta, y, u, h, \varepsilon), \quad y(\theta_0) = y^0 = X^0 \\ &u' = \varepsilon U(\theta, y, u, h, \varepsilon), \quad u(\theta_0) = u^0 = V^0 \\ &h' = \varepsilon H(\theta, y, u, h, \varepsilon), \quad h(\theta_0) = h^0 = Z^0 \end{aligned} \tag{2.13}$$

The functions Y, U and H are defined in the domain $\theta \geqslant \theta_0$, $y \in D_x$, $u \in D_V$, $h \in D_Z$ under consideration for sufficiently small $\varepsilon \in [0, \varepsilon_0]$ and have a uniform average with respect to θ. They will be smooth if the functions p, q and g are smooth. The right-hand sides in (2.13) are defined by analogy with (2.6) and (2.8) as follows:

$$\begin{aligned} &\varepsilon Y \equiv \varepsilon (I + \varepsilon P_y^{**})^{-1} (u - P_h^{**} g_*), \quad P_{y,h}^{**} \equiv p_{y,h}^{**'} \\ &\varepsilon U \equiv p_* - p_*^0 - \varepsilon (P_y^* - P_{0y}^*) Y - \varepsilon (P_h^* - P_{0h}^*) g_* + \varepsilon q_* \\ &\varepsilon H \equiv \varepsilon g_*, \quad P_{y,h}^* \equiv p_{y,h}^{*'}, \quad P_{0y,h}^* = p_{0y,h}^{*'} \end{aligned} \tag{2.14}$$

The notation p_*, q_*, and g_* means that the variables x, υ and z in p, q and g in (2.12) are replaced by

$$x = y + \varepsilon p^{**}(\theta, y, h), \quad \upsilon = \varepsilon u + \varepsilon[p^*(\theta, y, h) - \overset{*}{p}_0(y, h)], \quad z = h \tag{2.15}$$

The function p_*^0 is equal to p_* when $\varepsilon = 0$, i.e. $x = y$, $\upsilon = 0$, and $z = h$. It follows that $p_* - p_*^0 = O(\varepsilon)$ (see (2.8)). The functions p^*, p_*^0, and p^{**} can be constructed by the scheme (2.1)–(2.4), taking into account that p depends on $z = h$ in accordance with (2.12). We remark that the matrices with subscript y have dimensions $n \times n$, while those with subscript h have dimensions $n \times m$. In the first approximation with respect to ε the expressions for the functions in (2.14) are given below, see (3.6).

3. ANALYSIS OF THE SYSTEM IN THE FIRST APPROXIMATION

Consider the Cauchy problem (2.6), (2.8), (2.9) in the first approximation with respect to ε. Discarding terms $O(\varepsilon^2)$, we get

$$y' = \varepsilon \overset{*}{u}, \quad u' = \varepsilon P_x p^{**} + \varepsilon P_\upsilon (u + p^* - \overset{*}{p_0}) - \qquad (3.1)$$
$$- \varepsilon (P^* - \overset{*}{P_0}) u + \varepsilon q(\theta, y, u + p^* - \overset{*}{p_0}, 0)$$

where $P_{x,\upsilon}$, P^*, P_0^*, p^*, p_0^*, p^{**} are independent of u. Averaging with respect to the explicit argument θ and introducing the original time $t = \varepsilon\theta$, we obtain the following averaged system in the first approximation with respect to ε, in which t does not appear explicitly

$$y^{\cdot} = u, \; u^{\cdot} = r(y) + N(y)u + q_0(y, u), \quad y(t_0) = y^0, \quad u(t_0) = u^0 \qquad (3.2)$$
$$r(y) \equiv \langle P_x p^{**} \rangle + \langle P_\upsilon (p^* - \overset{*}{p_0}) \rangle, \quad N(y) \equiv \langle P_\upsilon \rangle$$
$$q_0(y, u) \equiv \langle q(\theta, y, u + p^* - \overset{*}{p_0}, 0) \rangle$$

For $n = 1$ the system can be studied by phase-plane methods [3]. The structure of (3.2) enables one to express it as a vector equation of the second order

$$y^{\cdot\cdot} = r(y) + N(y) y^{\cdot} + q_0(y, y^{\cdot}), \quad y(t_0) = y^0, \quad y^{\cdot}(t_0) = u^0 \qquad (3.3)$$

Comparison of (1.2) and (3.3) reveals that these equations differ in a non-trivial way.

Consider the stationary points of (3.2). One can easily establish that these points are

$$y^* = \operatorname{Arg}[r(y) + q_0(y, 0)], \quad u^* = 0 \quad (0 \equiv D_V) \qquad (3.4)$$

If an admissible root $y^* \in D_x$ exists, then the real parts of the characteristic exponents of the variational system determine whether the stationary solution of (3.2) or (3.3) is stable or unstable [1]. The corresponding characteristic equation can be expressed as

$$\Delta(\chi) = \det \begin{Vmatrix} -I\chi & I \\ r'(y^*) + q'_{0y}(y^*, 0) & N(y^*) + q'_{0u}(y^*, 0) - I\chi \end{Vmatrix} =$$

$$= \det \| I\chi^2 - (N(y^*) + q'_{0y}(y^*, 0))\chi - r'(y^*) - q'_{0y}(y^*, 0) \| = 0 \qquad (3.5)$$

Solving the algebraic equation (3.5) of degree $2n$ with respect to χ, we get

$$\chi_k = \operatorname{Arg}\Delta(\chi), \quad \chi_k = \sigma_k + i\gamma_k, \quad k = 1, 2, \ldots, 2n$$

The condition for asymptotic stability of the solution (3.4) of the averaged system (3.2) has the form $\sigma_k < 0$ $(k = 1, 2, \ldots, 2n)$. The solution is unstable if at least one of the numbers σ_k is greater than zero. The condition for asymptotic stability implies that the solutions of the original and averaged systems are close, in particular ε-close, to one another for $t \in [t_0, \infty)$, see [3, 4].

According to (2.12) and (2.14), in the first approximation with respect to ε the equations in terms of the osculating variables for the system (2.12) with variable parameters can be obtained by discarding the terms $O(\varepsilon^2)$ in (2.14)

$$
\begin{aligned}
&y' = \varepsilon(u - \overset{**}{P_h} \overset{0}{g_*}), \quad h' = \varepsilon \overset{0}{g_*} \\
&u' = \varepsilon P_x p^{**} + \varepsilon P_\upsilon(u + p^* - \overset{*}{p_0}) + \varepsilon(\overset{*}{P_y} - \overset{*}{P_{0y}})(u - \overset{**}{P_h} \overset{0}{g_*}) - \varepsilon(\overset{*}{P_h} - \overset{*}{P_{0h}})\overset{0}{g_*} + \varepsilon \overset{0}{q_*}
\end{aligned}
\tag{3.6}
$$

The symbols g_*^0, q_*^0 mean that one can set $\varepsilon = 0$ without loss of accuracy when g_* and q_* are smooth functions of ε. The averaged system will be an autonomic system of order $2n + m$. Its structure will be quite general. The stationary points and their stability can be determined in the usual way (see Section 4). Note that in a number of problems the time dependence of h can be determined explicitly in the form $h = h(t)$ or $h^{\cdot} = g_0(h)$. This may also mean that the time t appears twice in (1.1), namely, $F = F(t, \lambda t, X, X^{\cdot}; \lambda)$.

4. EXAMPLES

4.1. *A non-linear oscillator with variable parameters*

Consider a dynamical, and in particular a rotationally oscillatory system with one degree of freedom subjected to high-frequency interactions [2, 6]

$$
x^{\cdot\cdot} + Q(x, z) = P(\nu t, x, x^{\cdot}, z; \nu), \quad z^{\cdot} = Z(\nu t, x, x^{\cdot}, z; \nu) \tag{4.1}
$$

Here x is a scalar variable (a displacement or angle), z is the parameter vector (the mass-inertia characteristics, rigidity, the length of a pendulum, etc.). The initial values are given. The large parameter ν ($\nu \to \infty$) characterizes the frequency of the external kinematic interaction or force. When $P \equiv 0$ and $Z \equiv 0$ system (4.1) for x is conservative and can be solved completely in quadratures.

We introduce the fast time $\theta = \nu t$, which is the phase of the external interaction. When $x^{\cdot} = \nu x'$ and $z^{\cdot} = \nu z'$ system (4.1) can be transformed into the form (2.12)

$$
\begin{aligned}
&x'' = f(\theta, x, x', z, \varepsilon), \quad f \equiv \varepsilon^2(P - Q), \quad \varepsilon = \nu^{-1} \\
&z' = \varepsilon g(\theta, x, \varepsilon^{-1}x', z, \varepsilon), \quad g \equiv Z
\end{aligned}
\tag{4.2}
$$

According to Section 1 (see (1.2)), f or g may fail to be regular functions of $\upsilon = x'$. We assume that f defined in (4.2) has the structure (1.3)

$$
f \equiv \varepsilon p(\theta, x, \upsilon, z) + \varepsilon^2 q(\theta, x, \varepsilon^{-1}\upsilon, z, \varepsilon) \tag{4.3}
$$

where p and q are regular functions of their arguments. We note that g and q are regular functions of $V = \varepsilon^{-1}\upsilon$ and may also be smooth functions of υ.

We shall consider some special cases of (4.3). Let Q be an arbitrary function and let $P = P(\nu t, z; \nu)$ and $Z \equiv 0$. Then $z = \text{const}$ and the averaged variables y and u are governed by the autonomous system of equations

$$
y^{\cdot} = u, \quad u^{\cdot} = -Q(y, z) + a(z), \quad z = \text{const}
$$

In accordance with (4.2), it is assumed that $f = \varepsilon b(\theta, z) + \varepsilon^2 a(z) - \varepsilon^2 Q(x, z)$, where $\langle b(\theta, z) \rangle \equiv 0$. The system may have stationary points $u^* = 0$, $y^*(z) = \operatorname{Arg}(a - Q)$. If $\rho = -Q_y' > 0$ for some $y = y^*$, then $\chi_{1,2} = \pm\rho^{1/2}$ and the given point y^*, u^* is exponentially unstable. A critical situation arises when $\rho \leqslant 0$. Suppose that the system contains a dissipative interaction $-\varepsilon k(\theta, x, \varepsilon^{-1}\upsilon, z)\upsilon$ such that

$$k_0^* = \langle k(\theta, y, u + b^* - b_0^*, z)(u + b^* - b_0^*)\rangle \equiv \kappa(y, u, z)u + \delta(y, u, z),$$
$$\kappa_0 = \kappa(y^*, 0, z) \neq 0, \quad \delta(y, u, z) = O(u^2)$$

Then the averaged system will admit of the given stationary points y^*, u^*. If $\rho > 0$ and/or $\kappa_0 < 0$ for some point y^*, u^*. then the point is exponentially unstable. When $\rho < 0$ and $\kappa > 0$, the point is exponentially stable. The case when $\rho = 0$ and $\kappa_0 > 0$ is critical. Note that by (3.6), parametric excitation of system (1.4)–(4.3) is also possible in the case when $\langle Z \rangle \equiv 0$, i.e. $z = \text{const}$ in the first approximation with respect to ε.

4.2. *A simple pendulum of variable length*

Consider the equations of motion

$$Ml^2 x^{\cdot\cdot} + Mgl\sin x = -2MlDx^{\cdot} - K(l)x^{\cdot} + A(t, l) + B(t, \nu t, l; \nu) \tag{4.4}$$
$$l^{\cdot} = D(t, \nu t, x, x^{\cdot}, l; \nu)$$

of a simple pendulum of variable length taking into account the torques of viscous friction forces and external interactions. Here x is the angle of deflection from the lowest equilibrium position, M is the mass, l is the length, g is the acceleration due to gravity, K is the coefficient of friction, A and B are the slow and rapidly oscillating external torque components, and D is a function of the rate of variation of the length. The functions A, B and D may depend on t. We transform system (4.4), which is of the form (4.1), into the form (4.2), (4.3). We introduce the fast phase $\theta = \nu t$ and separate the first equation for $Ml^2\nu^2$ in (4.4) and the second equation for ν. We obtain

$$\begin{aligned} &x'' = \varepsilon l^{-2}[b - (2dl + k)x'] + \varepsilon^2 l^{-2}(a - gl\sin x) \equiv \varepsilon l^{-2} b + \varepsilon^2 l^{-2}[a - gl\sin x - (2dl + k)\varepsilon^{-1} x'] \\ &l' = \varepsilon d(t, \theta, x, \varepsilon^{-1} x', l, \varepsilon), \quad t' = \varepsilon, \quad \varepsilon = \nu^{-1} \\ &a = AM^{-1}, \quad b = \nu^{-1} BM^{-1}, \quad k = KM^{-1}, \quad d \equiv D \end{aligned} \tag{4.5}$$

Since $\langle b \rangle_0 \equiv 0$ by assumption, it follows from Section 2 that (4.5) can be reduced to the standard form (2.13), (2.14) by the replacement (2.15). In the first approximation with respect to ε we get

$$\begin{aligned} &y' = \varepsilon[u - (l^{-2} b^{**})'_l d_*^0 - l^{-2} b_t^{**}] \\ &u' = \varepsilon l^{-2}[a - gl\sin y - (2d_*^0 l + k)(u + l^{-2}(b^* - b_0^*))] - \\ &-\varepsilon[(l^{-2}(b^* - b_0^*))'_l d_*^0 + l^{-2}(b^* - b_0^*)'_t] \\ &l' = \varepsilon d_*^0 \equiv \varepsilon d(t, \theta, y, u + l^{-2}(b^* - b_0^*), l, 0) \end{aligned} \tag{4.6}$$

According to Section 3, the first approximation of the averaged system can be obtained by averaging over θ. The "slow" time t is regarded as a parameter. We observe that the average values of $b^* - b_0^*$ and $(b^* - b_0^*)'_t$, i.e. the last two terms in the equation for u' in (4.6), are certainly equal to zero. We shall consider some special cases of a, b, and d_*^0. Let $d_*^0 \equiv 0$ and let a and b be independent of t. Then, in terms of the original time t, we have

$$y^{\cdot} = u, \quad u^{\cdot} = l^{-2}[a(l) - gl\sin y - k(l)u], \quad l = l^0 \tag{4.7}$$

For $|r| \leqslant 1$ and $r = a(gl^0)^{-1}$ system (4.7) has stationary points $u^* = 0$, $y^* = \arcsin r$. The variational equations have one vanishing characteristic exponent, which corresponds to l. The other two are determined by the roots of the quadratic equation $\chi^2 + (l^0)^{-2} k\chi + g(l^0)^{-1}\cos y^* = 0$. For one of the numbers $y^* (\bmod 2\pi)$ we have $\cos y^* > 0$ and both roots have negative real parts. For the other number $\cos y^* < 0$, and one of the exponents is positive. It follows that for a pendulum of constant length ($d \equiv 0$ and $l = \text{const}$) with $|r(l)| < 1$ one of the states of equilibrium in the system under consideration is asymptotically stable, while the other one is unstable. The case $|r| = 1$ is critical.

We assume that d_*^0 in (4.6) does not depend explicitly on the phase θ. Then the averaged equations will be considerably simplified. In terms of the slow time t we have

$$\begin{aligned} &y^{\cdot} = u - d_*^0 (l^{-2} b_0^{**})'_l - l^{-2}(b_0^{**})'_t \\ &u^{\cdot} = l^{-2}[a - gl\sin y - (2d_*^0 l + k)u], \quad l^{\cdot} = d_*^0 \end{aligned} \tag{4.8}$$

Moreover, if a, b and d_*^0 do not depend explicitly on t, then the stationary points of (4.8) can be determined in the usual way. To fix our ideas, we take $d_*^0 = c(l_0 - l)$, where $c > 0$ and $l > 0$ are given constants. Then $l \to l_0$ exponentially with exponent $-c < 0$. The stationary points of (4.8) can be determined in the same way as those of (4.7), namely, $y^* = \arcsin r(l_0)$, $u^* = 0$, and $l^* = l_0$ if $|r(l_0)| \leqslant 1$. One of the characteristic exponents is equal to $-c$, the other two being the roots of the aforesaid quadratic equation with $l^0 = l_0$. The exponential stability and conditions of instability have been established before.

Let us consider how the motion of the pendulum is affected by rapid variations of the length. Let the functions $b = b(\theta)$ and $d_*^0 = d_*^0(\theta)$ be independent of t and l. Then, on averaging (4.6) over θ, we obtain

$$y^{\cdot} = u + 2l^{-3}\langle b^{**} d_*^0\rangle, \quad l^{\cdot} = \langle d_*^0\rangle \tag{4.9}$$

$$u^{\cdot} = l^{-2}(a - gl\sin y - ku)$$

Even in the case when $\langle d_*^0\rangle = O(l \simeq l^0)$ remains in the first equation for y, the term $2\langle b^{**} d_*^0\rangle$ is, in general, non-zero. In this case the system may have stationary points, which can be studied in the usual way. We remark that for the above model as well as for more general ones, problems concerned with the parametric control of oscillations and rotations by periodic variations of the length can be stated and studied approximately [6].

4.3. *A plastic model of the variation in pendulum length*

We consider a quasistationary model of the variations in the length l of a pendulum due to plastic deformations of the "thread" caused by tension [4], the magnitude of which we take to be $E = Mg\cos x + M/x^2$. We assume that the deformation rate is proportional to the tension, namely, $l^{\cdot} = \kappa E$, where the coefficient $\kappa > 0$ may depend on l. We assume that this relation holds for positive as well as negative values of E.

By (4.6), in the first approximation the averaged equations take the form

$$y^{\cdot} = u - \kappa E_0^*(l^{-2} b_0^{**})'_l, \quad l^{\cdot} = \kappa E_0^* \tag{4.10}$$

$$u^{\cdot} = l^{-2}[a - gl\sin y - (2\kappa E_0^* l + k)u] - 2\kappa l^{-3} M(2\langle \Delta b^{*2}\rangle u + l^{-2}\langle \Delta b^{*3}\rangle)$$

$$E_0^* = M(g\cos y + lu^2 + l^{-3}\langle \Delta b^{*2}\rangle), \quad \Delta b^* = b^* - b_0^*$$

Here we assume for simplicity that a and b are independent of t.

System (4.10) can be studied by numerical and qualitative methods. To determine the stationary points y^*, u^* and l^*, we shall consider the simplest case when $a = \text{const}$, $b_0^{**} = 0$, and $\langle \Delta b^{*3}\rangle = 0$. Then we obtain the relationships

$$u^* = 0, \quad a - gl\sin y = 0, \quad g\cos y + l^{-3}\langle \Delta b^{*2}\rangle = 0$$

The second equality constitutes the equilibrium condition and the third one means that the average length is constant, i.e. there is no tension. On eliminating l, we obtain the transcendental equation $\sin^3 y = -e\cos y$ for y, $e = a^3(g^2\langle \Delta b^{*2}\rangle)^{-1}$ being an arbitrary parameter. This equation can be reduced to a cubic equation of the form $\xi^3 - e^2(1-\xi) = 0$ with $1 \geqslant \xi \geqslant 0$ for the unknown $\xi = \sin^2 y$. The graphical analysis of the transcendental equation for y reveals that the interval $0 \leqslant y < 2\pi$ contains two roots $y_{1,2}^*$ such that $0 \leqslant y_1^* < \pi$ and $y_2^* = y_1^* + \pi$, independently of the value and sign of $e \neq 0$. As a result, we obtain two stationary points $y_{1,2}^*$, 0, $l_{1,2}^*$, from which to choose the point that corresponds to positive length $l_{1,2}^* = a(g\sin y_{1,2}^*)^{-1}$. Namely, we set $y = y_{1,2}^*$ $(\sin y_{1,2}^* \gtrless 0)$ if $a \gtrless 0$.

The problem of stability involves an analysis of the roots of the characteristic equation, which has a rather complex form. In the limiting case when $a = 0\,(e = 0)$ the roots are $y_1^* = 0$ and $y_2^* = \pi$. From the condition that l^* must be positive we find the admissible value $y^* = y_2^* = \pi$. Then $l^* = (\langle \Delta b^{*2}\rangle g^{-1})^{1/3}$ and the characteristic equation splits into the above quadratic and linear equations. The quadratic equation implies that the upper equilibrium state $y^* = \pi$ is exponentially unstable. This is also the case for small $a \neq 0$. We note that the upper equilibrium state can be made asymptotically stable by vibrations of the point of suspension [3, 4, 6], see below. The case when a is asymptotically large leads to the lateral equilibrium states $y_{1,2}^* = \pm 1/2\pi + O(e^{-1})$, which also turn out to be exponentially unstable.

4.4. *A pendulum with a vibrating point of suspension*

We will consider the model described in Section 4.3 under the assumption that the pendulum is subjected to kinematic perturbations due to planar oscillations $\xi = \xi(\nu t)$, $\eta = \eta(\nu t)$ of the point of suspension, rather than the torque $A+B$ of external forces about the axis. Here ξ is the horizontal displacement and η is the vertical displacement. The equations of motion of the type (4.2)–(4.4) take the form

$$\begin{aligned} &Ml^2 x^{\cdot\cdot} + Mgl\sin x = -2Ml\kappa E x^{\cdot} - Kx^{\cdot} - Ml\nu^2 W \\ &l^{\cdot} = \kappa E, \quad K = \mathrm{const}, \quad W = \xi'' \cos x + \eta'' \sin x \end{aligned} \tag{4.11}$$

On introducing the argument $\theta = \nu t$ and dividing by $Ml^2\nu^2$ and ν, we obtain a system of the form (4.5), in which we set

$$\begin{aligned} &p = l^{-1} W(\theta, x), \quad \xi'' = \varepsilon \xi_*'', \quad \eta'' = \varepsilon \eta_*'' \\ &q = -l^{-2}(gl\sin x + 2\kappa l E \varepsilon^{-1} \upsilon + k\varepsilon^{-1}\upsilon), \quad \upsilon = x' \\ &d = \kappa E \equiv \kappa M[g\cos x + l(\varepsilon^{-1}\upsilon)^2] \end{aligned} \tag{4.12}$$

In place of x and υ we introduce the variables y and u by (2.5). In the first approximation we obtain the system (see (3.6))

$$\begin{aligned} &y' = \varepsilon[u + l^{-2} S(\theta, y)\kappa E], \quad S \equiv \chi_* \cos y + \eta_* \sin y \\ &u' = \varepsilon l^{-2} (W_x')^* S - \varepsilon l^{-1} V_y'(u + l^{-2} S\kappa E) + \\ &+\varepsilon l^{-2} V\kappa E - \varepsilon l^{-2}[gl\sin y + (2l\kappa E + k)(u + l^{-1}V)] \\ &l' = \varepsilon\kappa M[g\cos y + l(u + l^{-1}V)^2], \quad V = S_\theta' \end{aligned} \tag{4.13}$$

The system of equations (4.13) is quite complex. After averaging, using the fact that S is periodic and the mean values of V and W with respect to θ are equal to zero, we obtain much simpler equations. To fix our ideas, suppose that the point of suspension undergoes only vertical harmonic oscillations $\eta_* = \eta_0 \sin\theta$ ($\xi_* \equiv 0$). We observe that after introducing the parameter $\varepsilon = \nu^{-1}$, η_* and η_0 will have the dimension of velocity by (4.12). In terms of the slow (original) time we obtain an averaged system of the form

$$\begin{aligned} &y^{\cdot} = u, \quad l^{\cdot} = \kappa M(g\cos y + lu^2 + \tfrac{1}{2} l^{-1}\eta_0^2 \sin^2 y) \\ &u^{\cdot} = -\tfrac{1}{2} l^{-2}\eta_0^2 \sin y\cos y - gl^{-1}\sin y - l^{-2}ku - \\ &-2\kappa M l^{-1} u(g\cos y + \tfrac{3}{2} l^{-1} u\eta_0^2 \sin^2 + lu^2) \end{aligned} \tag{4.14}$$

The stationary points of (4.14) can be determined from the equations $(u^* = 0)$

$$g\cos y + \tfrac{1}{2} l^{-1}\eta_0^2 \sin^2 y = 0, \quad (\tfrac{1}{2} l^{-1}\eta_0^2 \cos y + g)\sin y = 0$$

Since $y = 0$, π are not roots, after dividing the second equation by sin y we obtain the relationship $\sin^2 y = \cos^2 y$, from which to determine y. We obtain four values $y_j^* = \pi/4 + 1/2\pi(j-1)$ ($j = 1, 2, 3, 4$). It is interesting that the stationary values y_j^* are independent of the parameters of the system. The admissible value y_j^* can be determined from the condition $l_j^* > 0$. Since $l_j^* = -1/2\eta_{0_g}^{2\;-1} \cos y_j^*$, we take $y_2^* = 3\pi/4$ and $y_3^* = 5\pi/4$. Then $l_{2,3}^* = \eta_0^2 (2\sqrt{2}g)^{-1}$.

The values $y_{1,4}^*$ lead to negative $l_{1,4}^*$. Below, as above, we analyse the stability of the resulting states of equilibrium. One of them is usually exponentially unstable while the other one is stable.

We consider the limiting case when $\kappa = 0$, i.e. $l = \mathrm{const}$. Then we obtain the extensively studied [3, 4] system (4.14) for y and u. We have the stationary value $u^* = 0$, and y^* can be determined from the equation $\sin y(1/2\eta_0^2 \cos y + gl) = 0$. It follows that $y_1^* = 0$ and $y_2^* = \pi$, y_1^* being the asymptotically stable lower state of equilibrium. The upper state y_2^* is exponentially unstable if $gl > 1/2\eta_0^2$. It is asymptotically stable if $1/2\eta_0^2 > gl$. Then two exponentially unstable states $y_{3,4}^* = \pm\arccos(-2gl\eta_0^{-2})$ determined by the roots of the factor in brackets appear between the upper and the lower states y_2^* and y_1^* The case

$1/2\eta_0^2 = gl$ is critical. the value $y_1^* = 0$ corresponds to an asymptotically stable equilibrium. There is one zero characteristic exponent corresponding to $y_2^* = y_3^* = y_4^* = \pi$ (the other one is negative). A similar result can be obtained in the case when l varies according to the equation $l^{\cdot} = c(l_0 - l)$ or in a similar way.

In the conclusion we consider the case of arbitrary planar vibrations of the point of suspension when the length l is constant. System (4.13) implies the following averaged equations for y and u

$$y^{\cdot} = u, \quad u^{\cdot} = l^{-2}[\tfrac{1}{2}(\langle\xi_*'^2\rangle - \langle\eta_*'^2\rangle)\sin 2y - \langle\xi_*'\eta_*'\rangle\cos 2y - gl\sin y - ku] \tag{4.15}$$

The states of equilibrium of (4.15) are determined by the roots of the equations $(u^* = 0)$, which can be reduced to a fourth-degree algebraic equation in the unknown sin y. Using elementary trigonometric transformations, it can be represented as the transcendental relationship $\sin y = \rho\sin(2y + \varphi)$, where $\rho \geqslant 0$ and $0 \leqslant \varphi < 2\pi$ are constants determined by the coefficients of the equation. The latter can be studied and solved by graphical and numerical methods by constructing a one-parameter family of curves, for example, of the form $\rho^{-1} = \sin(2y + \varphi)/\sin y$, where $0 \leqslant y < 2\pi$ is the argument and $0 \leqslant \varphi < 2\pi$ is a parameter.

Let the vibrations be such that $\langle\xi'_*\eta'_*\rangle = 0$. This is a situation similar to that considered above. The quantities y_j^* can be determined from the equation $\sin y(1 + \alpha\cos y) = 0$, but here $\alpha = (\langle\eta'^2_*\rangle - \langle\xi'^2_*\rangle)(gl)^{-1}$ can be positive, negative, or zero. When $|\alpha| < 1$ there are two stationary points $y_1^* = 0$, which is asymptotically stable, and $y_2^* = \pi$, which is exponentially unstable. If $\alpha > 1$, then the two equilibrium states are stable and the additional points $y_{3,4}^* = \arccos(-\alpha^{-1})$ are unstable (exponentially). Furthermore, if $\alpha < -1$, then the lower and upper states $y_{1,2}^*$ will be unstable, while the lateral states $y_{3,4}^*$ will by asymptotically stable. When $\alpha = 1$, the lower equilibrium state y_1^* remains stable (as for $\alpha > 1$), while the upper state y_2^* does not (the critical case). Finally, if $\alpha = -1$, the upper state y_2^* is exponentially unstable and the lower state is critical.

Note that the equality $\langle\xi'_*, \eta'_*\rangle = 0$ holds, in particular, when $\xi'_* \equiv 0$ or $\eta'_* \equiv 0$. The case $\xi'_* \equiv 0$, in which $\alpha = \langle\eta'^2_*\rangle(gl)^{-1} > 0$, has been considered before. In the case of horizontal oscillations $(\eta'_* \equiv 0)$ we have $\alpha = -\langle\xi'^2_*\rangle(gl)^{-1} < 0$. In the general case this means that $\xi'_*(\theta)$ and $\eta'_*(\theta)$ are "orthogonal", for example, ξ'_* is an odd and η'_* is an even function of θ. To be specific, one can consider the motion of the point of suspension along an ellipse: $\xi_* = \xi_0\cos\theta$ and $\eta_* = \eta_0\sin\theta$, where ξ_0 and η_0 are the semi-axes. Then $\langle\xi'_*\eta'_*\rangle = 0$ and the parameter $\alpha = 1/2(\eta_0^2 - \xi_0^2)(gl)^{-1}$ can take any values, see above.

REFERENCES

1. MALKIN I. G., *Some Problems in the Theory of Non-linear Oscillations.* Gostekhizdat, Moscow, 1956.
2. AKULENKO L. D., On some rotationally oscillatory systems subjected to high-frequency perturbations. *Zh. Vychisl. Mat. Mat. Fiz.* **8**, 5, 113–1139, 1968.
3. BOGOLYUBOV N. N. and MITROPOL'SKII Yu. A., *Asymptotic Methods in the Theory of Non-linear Oscillations.* Nauka, Moscow, 1974.
4. VOLOSOV V. M. and MORGUNOV B. I., *The Method of Averaging in the Theory of Non-linear Oscillatory Systems.* Izd. Mosk. Gos. Univ, Moscow, 1971.
5. ZHURAVLEV V. F. and KLIMOV D. M., *Applied Methods in the Theory of Oscillations.* Nauka, Moscow, 1988.
6. AKULENKO L. D., *Asymptotic Methods of Optimal Control.* Nauka, Moscow, 1987.

Translated by T.J.Z.